Convergence of approximate solutions of conservation laws

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Abstract I

n this paper we consider convergence of approximate solutions of conservation laws. We start with an overview over the historical developments since the 1950s, and the analytical tools used in this context. Then we present some of our own results on the convergence of numerical approximations, discuss recent related work and open problems.

1 Exact and approximate solutions of conservation laws

Hyperbolic conservation laws are used widely to model the local conservation of physical quantities like density, momentum and energy. Applications include gasdynamics, ideal magnetohydrodynamics, shallow water and traffic flows. The equations take the form

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0, \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^d, \tag{1}$$

where $u(t,x) := (u_1(t,x), \ldots, u_m(t,x))$ is the m-vector of conservative variables with initial data $u(0,\cdot) =: u_0$, and $f(u) = (f_1(u), \ldots, f_d(u))$ is the flux function. It is well known that for nonlinear fluxes, even for smooth initial data, the solution of (1) may cease to exist in the classical sense due to the

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formation of shocks in finite time (cf. Riemann 1859 [43]). Therefore it is necessary to consider weak solutions, i.e. functions u that are bounded and satisfy (1) in the distributional sense. Existence of weak solutions for general systems of conservation laws is largely unknown, and weak solutions are not unique. However, for scalar equations (with $m = 1$) the Cauchy problem is well understood.

As usual, a possible strategy to establish existence of weak solutions of the conservation law (1) is first to regularize the problem and obtain a sequence of approximate solutions $\{u^\epsilon\}_\epsilon$ for $\epsilon > 0$, then to show that a subsequence of u^ϵ converges to a limit function u which is a weak solution of (1). The regularization mostly used is the vanishing viscosity method, where u^ϵ solves the parabolic problem $\partial_t u^\epsilon + \nabla \cdot f(u^\epsilon) = \epsilon \Delta u^\epsilon$. Since in general exact solutions of conservation laws are not known, approximations through numerical computations are of utmost importance in the applications. As the grid size tends to zero, again a sequence of approximate solutions is created, and then a convergence analysis is important to make sure that the results are reliable. We say more on that issue in Section 3.

We will assume that the sequence $\{u^\epsilon\}_\epsilon$ of approximate solutions is uniformly bounded in $L^\infty([0,T] \times \mathbb{R}^d)$. For the vanishing viscosity method and for many numerical schemes this can be obtained from a (discrete) maximum principle or an invariant region argument. Then it is possible to extract a subsequence which converges weak* to some limit function $u \in L^\infty([0,T] \times \mathbb{R}^d)$. But in order to prove that u is a weak solution of (1), it is necessary to show that the weak limit of $f(u^\epsilon)$ coincides with $f(u)$. For a weakly converging sequence u^ϵ and a nonlinear flux f this is not true in general because oscillations may occur. A sufficient condition, however, would be strong L^1_{loc}-convergence. So the question is how this additional information can be obtained.

We already mentioned that for a general nonlinear hyperbolic conservation law there may exist many different weak solutions corresponding to the same initial data. In order to ensure uniqueness one imposes an additional condition, often called an entropy condition because of its analogy with the second law of thermodynamics, which selects the physically relevant weak solution out of all possible ones. For scalar conservation laws several related formulations are in use. We give the one that relies on convex entropies.

We call a pair of functions $\eta\colon \mathbb{R}^m \to \mathbb{R}$ and $q\colon \mathbb{R}^m \to \mathbb{R}^d$ an entropy-entropy flux pair if η is convex and if the compatibility relations

$$q_k'(u) = \eta'(u) f_k'(u) \quad \text{for all } u \in \mathbb{R}^m \text{ and } k = 1, \dots, d \tag{2}$$

hold. Then the entropy condition reads as follows: Out of all weak solutions of the conservation law (1) given above, we only select those satisfying

$$\frac{\partial \eta(u)}{\partial t} + \nabla \cdot q(u) \le 0 \quad \text{in distributional sense} \tag{3}$$

for all entropy-entropy flux pairs (η, q). We also assume that the initial entropy is bounded: $\int_{\mathbb{R}^d} \eta(u_0)\,dx < \infty$. These solutions are called weak entropy solutions. For scalar conservation laws all convex functions η can serve as entropies, and it turns out that weak entropy solutions are unique. This is exceptional. For systems of conservation laws there often exists only a limited number or even only one single (the physical) entropy, because the corresponding compatibility relations are much more restrictive. The entropy inequalities provide crucial estimates in the convergence analyis for $\{u^\epsilon\}_\epsilon$, and the lack of enough entropy-entropy flux pairs is the main reason why the existence theory for systems of conservation laws is not as well developed as for the scalar case. For more on the entropy condition we refer to Lax [30] and Liu [34]. Further information on hyperbolic conservation laws can be found in Lax [29], Smoller [46] and Dafermos [12].

Our paper is organized as follows. In Section 2, we give an overview over the developments in the theory of conservation laws since the 1950s, and the analytical tools used to study the convergence of approximate solutions. These include classical compactness arguments, approximation theory, weak convergence methods (compensated compactness and measure-valued solutions), and the kinetic approach. In Section 3, we present some of our own work on the convergence of numerical approximations. Finally, in Section 4, we discuss recent related work and open problems.

2 Historical remarks on compactness arguments

In this section we give an overview over some analytical tools used to study convergence of sequences of approximate solutions for conservation laws.

2.1 Regularity estimates and compactness

The classical approach to prove strong L^1_{loc}-convergence of a sequence $\{u^\epsilon\}_\epsilon$ of functions is to prove that the u^ϵ possess some positive regularity, i.e. the sequence is uniformly bounded in some appropriate function space. If the function space is BV, the space of functions with bounded variation, then strong compactness in L^1_{loc} follows from Helly's Theorem.

In 1957 Ladyzhenskaya [27] and Oleinik [39], both former students of Petrovskii in Moscow, independently published results on a scalar conservation law with strictly convex flux in one space dimension. The entropy condition they used to select the physical solution (today known as "condition E") can be interpreted as a one-sided Lipschitz condition: For any time $t > 0$, the solution $u(t, \cdot)$ is Lipschitz continuous where it is increasing (in x). There is no restriction where it is decreasing. In particular, decreasing jump

discontinuities are allowed. From this a BV bound can be derived. We would like to remark that the one-sided Lipschitz continuity was already observed by E. Hopf [20] in his pioneering paper on the Burgers equation, but he did not use it systematically as selection criterion. While Ladyzhenskaya and Oleinik both studied the vanishing viscosity limit, Oleinik together with her student Vvedenskaya [54, 39] also proved convergence of the Lax-Friedrichs scheme [28]. In 1963, Oleinik's paper was translated into English, while Ladyzhenskaya and Vvedenskaya's work was not. Oleinik's contribution is also discussed in full detail in Smoller's famous book [46]. This may explain why condition E is nowadays usually attributed to Oleinik.

The multidimensional scalar problem was studied by Conway and Smoller [9] and Vol'pert [53] in the 1960s, and by Kruzkov [25] in 1970 who proved existence and uniqueness of weak entropy solutions for the scalar conservation law. Kruzkov's main observation was that the solution operator of the scalar conservation law is an L^1-contraction: He used the special convex entropies

$$\eta(u,k) := |u-k| \quad \text{and} \quad q(u,k) := \operatorname{sgn}(u-k)|f(u)-f(k)| \tag{4}$$

(nowadays known as Kruzkov entropies), where $k \in \mathbb{R}$ is a real parameter, and a "doubling of variables"-argument to conclude that

$$\int_{\mathbb{R}^d} |u(t,x)-v(t,x)|dx \le \int_{\mathbb{R}^d} |u_0(x)-v_0(x)|dx \tag{5}$$

for all times $t > 0$. Here u and v are weak entropy solutions of (1) corresponding to initial data u_0 and v_0, respectively. Uniqueness of weak entropy solutions of (1) is an immediate consequence of this estimate. And since the solution operator of (1) is invariant under translations in x, we can apply (5) also to $v(t,x) := u(t,x+h)$ for any $h \in \mathbb{R}^d$ and obtain from this a BV-bound $\|u(t,\cdot)\|_{BV(\mathbb{R}^d)} \le \|u_0\|_{BV(\mathbb{R}^d)}$. An analogous estimate for temporal regularity can then be derived using (1). So Helly's theorem can be applied.

Let us briefly mention that there is also a well-developed theory for systems of conservation laws in one space dimension relying on BV bounds. This theory started with the work of Glimm [17] in 1965 who introduced a numerical scheme that produces approximate solutions with bounded variation. Later on, Bressan and his coworkers extended Glimm's ideas considerably and set up a very powerful framework for studying existence and uniqueness of systems of conservation laws in 1-d. We refer to Bressan [3].

BV bounds also play a major role in the design of higher order accurate numerical algorithms. In 1983, Harten [19] introduced the class of TVD (total variation diminishing) schemes and proved their convergence in the linear case. Osher [40] introduced the class of "E-schemes", the most general class of schemes which satisfy a discrete entropy inequality, and proved convergence of second order accurate semidiscrete TVD schemes for nonlinear scalar conservation laws. Convergence of a second order accurate fully discrete scheme was proved by Osher and Tadmor in 1988 [41].

2.2 Approximation theory

Approximation theory in the context of scalar conservation laws means to estimate the distance between an approximate solution (at some time $t > 0$, say) and the uniquely defined weak entropy solution which is already known to exist. This approach gives error estimates and convergence rates. Applied to numerical schemes it tells you how close an approximate solution is to the exact one.

The first who gave error estimates for approximate solutions of scalar conservation laws in several space dimensions was Kuznetsov [26], who studied the vanishing viscosity method and the Lax-Friedrichs scheme. His analysis relies on the L^1-contraction property of the solution operator of (1) we discussed in the previous section. In fact, it is possible to obtain an estimate like (5) even if v is not an exact, but only a (suitable) approximate solution. Then additional terms measuring the approximation error enter on the right hand side which can be controlled if the initial data is in BV. More precisely, one can estimate the L^1-distance between the weak entropy solution u and the approximate solution u^ϵ at a given time t in terms of the distance and BV-norm of the initial data, the residual etc. These estimates depend on ϵ (which could be the gridsize in numerical schemes, for example), and the convergence rate proved by Kuznetsov is $\epsilon^{1/2}$.

Kuznetsov's ideas were used later on to give convergence rate estimates for more sophisticated numerical schemes, as well. We refer to Sanders [45], Vila [52], Cockburn, Coquel and LeFloch [7], Noelle [37], and Cockburn, Gremaud and Yang [8]. We also refer to Bouchut and Perthame [1] who reformulated Kruzkov and Kuznetsov's approximation theory in a general, versatile form.

We also mention that recently, Tadmor [50] and coworkers developed a somewhat different approach to proving error estimates and convergence rates for scalar conservation laws in 1-d with strictly convex flux, the so-called Lip'-theory. The idea is to measure the distance between the exact and the approximate solution not in the L^1-norm, but in the much weaker topology of $\mathrm{Lip}'(\mathbb{R})$. This is the topological dual of the space of Lipschitz continuous functions. The approach can also give pointwise error estimates: The theory shows that the convergence rate at some given point $x \in \mathbb{R}$ only depends on the regularity of the exact solution u in a small neighborhood around x.

2.3 Weak convergence methods

For modern numerical schemes using unstructured grids it may be too hard or even impossible to prove positive regularity in terms of boundedness of a sequence $\{u^\epsilon\}_\epsilon$ in appropriate function spaces. So the classical approach of proving compactness in L^1_{loc} by using Helly's theorem or Sobolev embeddings breaks down. There are other tools, which run under the name of Weak

Convergence Methods, that can prove strong compactness without showing regularity first. These methods are specialized in the sense that they rely heavily on the structure of the problem at hand.

In 1954, Lax [28] proved compactness in L^1 of the solution operator of a one-dimensional, not necessarily strictly convex, scalar conservation law using a weak topology. More than twenty years later, Tartar [51] and Murat [35] introduced an even more powerful method, applicable also for systems of conservation laws, the compensated compactness theory. A typical result is the div-curl-Lemma: One assumes that two sequences of vector-valued functions $\{U^\epsilon\}_\epsilon$ and $\{V^\epsilon\}_\epsilon$ are bounded in $(L^2_{loc}(\mathbb{R}^n))^N$ and that the divergence of U^ϵ and the curl of V^ϵ are both precompact in $H^{-1}_{loc}(\mathbb{R}^n)$. Then there exist subsequences such that

$$U^\epsilon \cdot V^\epsilon \longrightarrow U \cdot V \quad \text{in distributional sense.} \tag{6}$$

Note that the assumptions on the differentiability of U^ϵ and V^ϵ are not sufficient to obtain strong compactness in $L^2_{loc}(\mathbb{R}^n)$ (from which (6) would follow trivially). But in combination they give enough information to have weak continuity of the scalar product.

The div-curl-Lemma has been used by Tartar to study the convergence of sequences of approximate solutions $\{u^\epsilon\}_\epsilon$ of the scalar conservation law (1) in one space dimension. Compensated Compactness arguments can also be used to study certain systems of conservation laws in 1-d. As an example, we refer to the work of DiPerna [13] and Chen [5] on isentropic Euler equations.

Another weak convergence method is DiPerna's theory of measure-valued solutions for multidimensional scalar conservation laws. As mentioned before, we assume that the sequence $\{u^\epsilon\}_\epsilon$ of approximate solutions is uniformly bounded in L^∞ and extract a subsequence converging weak* in L^∞. Then the sequence converges strongly in L^1_{loc} if and only if no oscillations occur, i.e. if there is convergence pointwise a.e. The concept of measure-valued solution was first introduced by Tartar [51] in 1975, who used Young measures to describe the oscillations occuring in $\{u^\epsilon\}_\epsilon$. A Young measure associated to $\{u^\epsilon\}_\epsilon$ is a weakly measurable mapping ν from $[0,T] \times \mathbb{R}^d$ into the space of probability measures $\mathrm{Prob}(\mathbb{R})$, such that for all continuous functions f we have $f(u^\epsilon) \to <\nu, f>$ in distributional sense, where the pairing $<\nu, f>$ is given by

$$<\nu, f>(t,x) := \int_{\mathbb{R}} f(\lambda)\, d\nu_{(t,x)}(\lambda).$$

So the distributional limit of any nonlinear function of u^ϵ can be written down using one single ν. A Young measure is called a measure-valued solution of the Cauchy problem (1) if the following identity holds

$$\frac{\partial}{\partial t} <\nu, \mathrm{id}> + \nabla \cdot <\nu, f> = 0 \quad \text{in distributional sense.} \tag{7}$$

Consistency with the entropy condition then means: For all convex η

$$\frac{\partial}{\partial t} <\nu, \eta> + \nabla \cdot <\nu, q> \leq 0 \quad \text{in distributional sense,} \tag{8}$$

where q is the corresponding entropy flux.

Note that a measure-valued solution is even weaker than a weak (distributional) solution. One can show that for any bounded sequence of approximate solutions $\{u^\epsilon\}_\epsilon$ there exists a Young measure ν and a subsequence converging to it in an appropriate sense. That ν is a measure-valued solution of (1) then follows from the consistency of the approximation.

One has $<\nu, \mathrm{id}> = u$, where u is the distributional limit of $\{u^\epsilon\}_\epsilon$. But in general $<\nu, f> \neq f(u)$. Equality holds for arbitrary nonlinear f if and only if the Young measure reduces to a Dirac measure, i.e. if ν is of the form $\nu_{(t,x)} = \delta_{u(t,x)}$ with δ_u the Dirac distribution on $\mathbb{R}$ centered at u. This is equivalent to saying that the sequence $\{u^\epsilon\}_\epsilon$ converges strongly in L^1_{loc}.

The following result is due to DiPerna [14] (see also Szepessy [47])

Theorem 1. *Assume that initial data $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ is given and that there is exists a Young measure ν satisfying the following properties*
(i) the function $(t,x) \mapsto <\nu_{(t,x)}, |\mathrm{id}|>$ is in $L^\infty([0,T], L^1(\mathbb{R}^d))$,
(ii) ν is a measure-valued solution of (1),
(iii) ν is consistent with the entropy condition,
(iv) ν assumes the initial data u_0 in the following sense

$$\lim_{t \to 0} \frac{1}{t} \int_0^t \int_{\mathbb{R}^d} <\nu_{(s,x)}, |\mathrm{id} - u_0(x)|> ds\, dx = 0. \tag{9}$$

Then the Young measure reduces to a Dirac measure, i.e. $\nu_{(t,x)} = \delta_{u(t,x)}$ for a.e. $(t,x) \in [0,T] \times \mathbb{R}^d$, where u is the unique entropy solution of (1).

Assumption (iv) says that the initial data must be attained in a stronger sense than just the sense of distributions. It excludes the occurence of oscillations in the sequence of approximate initial data $\{u_0^\epsilon\}_\epsilon$. Then condition (iii) assures that no oscillations can develop at later times: the only oscillations that can exist in the sequence $\{u^\epsilon\}_\epsilon$ are those transported into the system from the initial data. If the initial data converge strongly, so does $\{u^\epsilon\}_\epsilon$ at any later time.

Theorem 1 was used to prove convergence of the streamline diffusion shock-capturing method by Szepessy [48], finite difference methods by Coquel and LeFloch [10], spectral viscosity approximations by Chen, Du and Tadmor [6] and finite volume schemes on unstructured polygonal grids by Kröner and Rokyta, [24], Kröner, Noelle and Rokyta [22], Noelle [36], and others. Rohde [44] extended the method to weakly coupled systems of conservation laws.

2.4 Kinetic formulation and velocity averaging

The kinetic formulation was introduced by Lions, Perthame, and Tadmor [32]. They show that there is a one-to-one correspondence between weak entropy solutions of the scalar conservation law (1) and solutions of a linear transport equation with source term, for which a certain nonlinear constraint holds. More precisely, one considers a "density-like" function ρ depending on $(t,x) \in [0,t] \times \mathbb{R}^d$, and on an additional variable $v \in \mathbb{R}$, which is a solution of the transport equation

$$\frac{\partial \rho}{\partial t} + f'(v) \cdot \nabla \rho = \frac{\partial m}{\partial v} \quad \text{in distributional sense.} \tag{10}$$

Here m is a nonnegative bounded measure. Equation (10) is supplemented with an assumption on the structure of ρ. If $\chi \colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\chi(v|\alpha) := \begin{cases} +1 & \text{if } 0 < v < \alpha \\ -1 & \text{if } \alpha < v < 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

for $\alpha \in \mathbb{R}$, then ρ should have the form

$$\rho(t,x,v) = \chi(v|u(t,x)) \tag{12}$$

for some scalar function u. If ρ is a solution of the kinetic equation (10) and satisfies the nonlinear constraint (12), then the function u is the unique weak entropy solution of (1). Vice versa, if u is the unique entropy solution of (1), then there exists a nonnegative bounded measure m such that the function ρ defined by (12) is a solution of the kinetic equation (10).

One big advantage of the kinetic formulation for scalar conservation laws is the possibility to apply velocity averaging lemmas to obtain regularity and compactness for (sequences of approximate) weak entropy solutions of (1). One can show that the macroscopic quantity u has more regularity than the ρ whose v-average it is, see Golse, Lions, Perthame and Sentis [18] and DiPerna, Lions and Meyer [15].

In Westdickenberg and Noelle [55] we used the kinetic formulation together with the velocity averaging technique to prove the convergence of a class of finite volume schemes for scalar conservation laws in several space dimensions. We will discuss this result in more detail in Section 3.3 below.

We remark that already in 1984, Brenier [2] constructed and analyzed an approximate evolution operator for (1) based on a kinetic decomposition of u. In 1998 Perthame [42] used the kinetic formulation to study again uniqueness of weak entropy solutions and error estimates in the sense of Kuznetsov's approximation theory. It is also possible to give a kinetic formulation for certain systems of conservation laws, e.g. the isentropic Euler equations in 1-d. In Lions, Perthame, Tadmor [33] and Lions, Perthame, Souganidis [31]

this kinetic formulation was used together with compensated compactness to prove existence of weak entropy solutions, thereby extending DiPerna [13] and Chen's [5] results mentioned above.

3 Some convergence results for finite volume schemes

In this section we give an overview over some of our own results on the convergence of finite volume schemes. These schemes define approximate solutions which are piecewise polynomial on a given polygonal grid of $\mathbb{R}^d$, and in general discontinuous at the cell interfaces. Integrating the conservation law (1) over a cell K with faces $e \subset \partial K$ from time t^n to time $t^{n+1} = t^n + \Delta t$ gives the following update formula for the cell average u_K^{n+1}:

$$u_K^{n+1} = u_K^n - \frac{\Delta t}{|K|} \sum_{e \subset \partial K} |e|\, g_{K,e}^n. \tag{13}$$

Here $g_{K,e}^n$ is a numerical flux which is consistent with $f(u) \cdot n_{K,e}$, where $n_{K,e}$ is the outward unit normal of face e. This flux has to be conservative, satisfy some upwinding property (e.g. an E-flux), and it may be higher order accurate in space and time, see e.g. [22, 36]. We consider explicit time discretizations, which means that the fluxes $g_{K,e}^n$ can be computed directly from the cell averages $u_{K'}^n$ in a neighborhood of K. Such discretizations are only stable for timesteps Δt which are so small that waves originating from face $e = K \cap K'$, where the approximation is discontinuous, do not cross the neighboring cells K and K' completely during one timestep. If one imposes this CFL-condition (named after Courant, Friedrichs and Lewy [11]), then it is possible to prove a discrete maximum principle.

3.1 Discrete entropy inequalities

The discrete maximum principle is enough to assure that a subsequence converges weak* to some limit function, but it does not guarantee that this limit is a weak solution of the conservation law. The crucial additional estimate needed to prove convergence is a discrete entropy inequality of the form

$$\eta(u_K^{n+1}) - \eta(u_K^n) + \frac{\Delta t}{|K|} \sum_{e \subset \partial K} |e|\, G_{K,e}^n \le C h^{2\alpha}. \tag{14}$$

Here h is the maximal diameter of a cell of the computational grid, $\alpha \in (\frac{1}{2}, 1]$ is a constant depending on the triangulation and the polynomial reconstruction, the constant C depends on the entropy η and its derivatives, and $G_{K,e}^n$

is a numerical entropy flux which is consistent with the entropy flux $q \cdot n_{K,e}$ and is closely related to the numerical flux $g^n_{K,e}$.

In [22] we proved an entropy inequality of this type for a class of higher order schemes based on the Lax-Friedrichs and the Engquist-Osher schemes, and in [36] we could treat schemes which extend Godunov's exact Riemann solver to higher order accuracy (see [22, 36] for the definitions and the original references).

There are several points in the derivation of estimate (14) which are worth recalling. In [36] we first rewrite the update (13) as a convex combination of one-dimensional contributions. This is natural, since the numerical fluxes $g^n_{K,e}$ approximate one-dimensional fluxes $f \cdot n_{K,e}$ propagating in the normal direction to the cell-faces. Adapting the weights of the decomposition to the local wave speeds, our analysis admits larger time steps than previous work, even for the classical first order schemes in 1-d [49].

Next we use Tadmor's decomposition [49] of any E-flux as a convex combination of Lax-Friedrichs and Godunov's fluxes. If one uses a piecewise constant reconstruction, corresponding to a first order accurate discretization, then the discrete entropy inequality with $C = 0$ follows naturally from the upwinding properties and the CFL-condition. For higher order polynomial reconstructions, we have to generalize Harten's TVD conditions discussed in Section 2.1 above to multidimensional reconstructions. The estimate for the higher order Lax-Friedrichs numerical flux follows readily, but for the higher order Godunov's flux, which yields the best resolution of shock discontinuities, the structure of the Riemann solution has to be exploited in full detail, see [36]. Once the one-dimensional discrete entropy inequalities are obtained, convexity arguments yield the multidimensional inequality (14).

3.2 Convergence and error estimates via L^1-contraction

Equipped with the discrete maximum principle and entropy inequality (14), we can now prove convergence and error estimates. In [22] we gave the first prove of convergence for higher order accurate finite volume schemes on unstructured grids for initial data which are merely in L^∞, using DiPerna's theory of measure-valued solutions. Subsequently, Kröner and collaborators designed practical algorithms along the lines of our convergence results.

In closely related work, Cockburn, Coquel and LeFloch [7] used Kuznetsov's approximation theory to prove convergence and error estimates for BV initial data in 1994. In [37], we could generalize these error estimates, and simplify their proof, using our discrete entropy inequalities.

Even though inequality (14) is the crucial estimate, careful work is required to finish the convergence proof. In this process, many papers (including [22]) assume that the triangulation is regular, meaning that the ratio of the outer

diameter to the inner diameter of a cell is bounded above. In [36], we removed that restriction and treated grids where the cells may become flat in the limit as $h \to 0$. In [37], we quantified the influence of irregular grids on the convergence rate. An example in [36] shows that the rate with which the grid may degenerate is optimal.

3.3 Convergence via kinetic formulation

In Westdickenberg and Noelle [55] we gave a new convergence proof for sequences of approximate solutions of the scalar conservation law, produced by higher order finite volume schemes on unstructured grids in several space dimensions. We did not use the theory of measure-valued solutions, but relied on the kinetic formulation and velocity averaging lemmas instead. The key tools in our analysis were once more the discrete entropy condition (14) and the following compactness theorem.

Theorem 2. *Let $1 < p \leq 2$ and $0 < \gamma < 1$. Fix a compact set $K \subset \mathbb{R}$ and assume that there are sequences $\{\rho^\epsilon\}_\epsilon, \{m^\epsilon\}_\epsilon$, and $\{\pi^\epsilon\}_\epsilon$ uniformly bounded in $L^p(\mathbb{R}^{d+1} \times K), L^1(K, \mathcal{M}(\mathbb{R}^{d+1}))$, and $L^1(K, C^*_\gamma(\mathbb{R}^{d+1}))$, resp., such that*

$$\frac{\partial \rho^\epsilon}{\partial t} + f'(v) \cdot \nabla \rho^\epsilon = \frac{\partial m^\epsilon}{\partial v} + \pi^\epsilon \quad \text{in distributional sense} \tag{15}$$

($(t,x) \in \mathbb{R}^{d+1}$, $v \in K$). If the following nondegeneracy condition holds

$$\sup_{(\tau,\xi)\in\mathbb{R}^{d+1}} \operatorname{meas}\bigl\{v \in \Lambda : \tau + f'(v) \cdot \xi = 0\bigr\} = 0, \tag{16}$$

then the sequence $\{z^\epsilon\}_\epsilon$ belongs to a compact subset of $L^1_{loc}(\mathbb{R}^{d+1})$, where

$$z^\epsilon := \int_K \rho^\epsilon(\cdot, v)\, dv. \tag{17}$$

Here, $\mathcal{M}(\mathbb{R}^{d+1})$ is the space of bounded Radon measures, and $C^*_\gamma(\mathbb{R}^{d+1})$ is the topological dual of the space of Hölder continuous functions. This theorem is a variant of the compactness theorem in [32]. Assumption (16) means that the flux function must be nonlinear enough. The regularizing effect stated in Theorem 2 does not exist for linearly degenerate problems (e.g. for advection equations). Note that the m^ϵ on the right hand side of (15) is a bounded measure, as we should expect from the kinetic formulation (10). The quantity π^ϵ is an error term (e.g. the numerical residual). It is measured in a function space with negative regularity, hence in a rather weak topology.

To prove Theorem 2 we decompose z^ϵ into two parts and show that one part can be made arbitrarily small in L^1_{loc} using the nondegeneracy of the flux (16), while the other part has some positive regularity in terms of Sobolev

(in fact Besov) spaces and is therefore strongly compact in L^1_{loc}. Applying Theorem 2 to numerical approximations we obtain the strong compactness needed to pass to the limits in nonlinear quantities as explained in Section 1.

4 Related work and open problems

In [23, 38] Kröner and Ohlberger derived a posteriori error estimates based on Kuznetsov's approximation theory and recent related work of Gallouet, Herbin, Chainais-Hillairet et al. [4]. Ohlberger [38] developed a fully adaptive, implicit finite volume scheme for scalar convection-reaction-diffusion equations based on these estimates.

In a recent preprint, Hwang and Tzavaras [21] used the kinetic formulation together with the velocity averaging technique to study the convergence of approximate solutions of scalar conservation laws that are produced by either a relaxation method or by a diffusion-dispersion approximation.

To conclude this note, we would like to sketch some important open problems concerning the convergence of numerical approximations. The convergence rate of $h^{1/2}$ proved by Kuznetsov for the Lax-Friedrichs scheme is only optimal for first order approximations of discontinuous solutions of linear advection equations. The rate obtained so far for unstructured grids is only $h^{1/4}$. A first attempt to overcome this barrier may be found in [8]. For strictly convex scalar conservation laws one expects a rate of $h \log h$, but this conjecture is so far only backed up by numerical experiments. For schemes which are formally higher order accurate one expects higher convergence rates away from discontinuities. For strictly convex scalar conservation laws a result in this direction is proven in [16]. It is also shown there that for systems of conservation laws, characteristics of one family crossing a numerical shock layer belonging to a different family may carry first order errors into the smooth postshock region, so the convergence rate deteriorates there even for higher order schemes. In ongoing work G. Kreiss et al. are beginning to design special numerical schemes which attempt to filter out this pollution effect.

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